1. State why the following is true or provide a counter example.

   (a) If \( S \) is a subset of a vector space \( V \), then the \( \text{span}(S) \) equals the
       the intersection of all subspaces of \( V \) that contain \( S \).

   (b) Every vector space has a finite basis.

   (c) Every subspace of a finite dimensional vector space is finite dimensional.

   (d) If a \( n \times n \) matrix has rank \( n \) it is invertible.

2. Find a basis for the vector space of matrices of size \( 3 \times 2 \).

3. Let
   \[
   A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.
   \]
   Find \( \det(A - tI) \) where \( t \in \mathbb{R} \) and \( I \) is the identity matrix.

4. For what real numbers \( t \) does the following system have a unique solution?
   \[
   \begin{align*}
   x_1 + tx_3 &= 1 \\
   2x_1 + x_2 - x_3 &= 1 \\
   x_2 + 2x_3 &= -3
   \end{align*}
   \]

5. Construct a \( 3 \times 5 \) matrix with rank 2.

6. A matrix is a projection matrix if \( P^2 = P \). Find a nontrivial \( 3 \times 3 \) projection matrix and show that it satisfies the definition. Let

7. Find a basis for the column space and null space of the matrix
   \[
   A = \begin{bmatrix} 2 & -2 & -3 & 0 \\ 3 & -3 & -2 & 5 \\ 1 & -1 & -2 & -1 \end{bmatrix}.
   \]

8. Is \( \{1 + 2x + x^2, 3 + x^2, x + x^2\} \) a basis for \( \mathbb{P}^2 \)? Show your work.

9. Compute the rank of
10. Let $V = \mathbb{R}^3$, $T(a, b, c) = (a + b, 2b - a, 2a + c)$, and $z = e_1$. Show that $\beta = \{z, T(z), T^2(z)\}$ forms a basis for $V$. We call $\beta$ a $T$-cyclic basis.

**Definition 1.** Let $T$ be a linear transformation from $V$ to $V$ and $W$ be a subspace of $V$. The subspace $W$ is $T$-invariant if $T(w) \in W$ for all $w \in W$.

11. Let $V = \mathbb{R}^3$, $T(a, b, c) = (a + b + c, a + b + c, a + b + c)$, and

$$W = \{(t, t, t) : t \in \mathbb{R}\}.$$ 

Is $W$ a $T$-invariant subspace of $V$?

12. Let $T$ be a linear transformation from $V$ to $V$. Show $\text{Null}(T)$ is $T$-invariant.