

Using Pattern Tasks to Develop Mathematical Understandings and Set Classroom Norms

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The capacity to reason algebraically is critical in shaping students' future opportunities and, as such, is a central theme of K–12 education (NCTM 2000). One component of algebraic reasoning is "the capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined functional rules" (Driscoll 1999, p. 2). Geometric pattern tasks can be a useful tool for helping students develop algebraic reasoning, because the tasks provide students with opportunities to build patterns with materials such as toothpicks or pattern blocks. These materi-

als help students "focus on the physical changes and how the pattern is being developed" (Friel, Rachlin, and Doyle 2001, p. 10). Such work might help bridge students' earlier mathematical experiences and lay the foundation for more formal work in algebra (English and Warren 1998; Ferrini-Mundy, Lappan, and Phillips 1997; NCTM 2000). Finally, the relationships between the quantities in pattern tasks can be expressed using symbols, tables, and graphs, as well as words. Thus, pattern tasks can also give students opportunities to make connections among representations—a key compo-

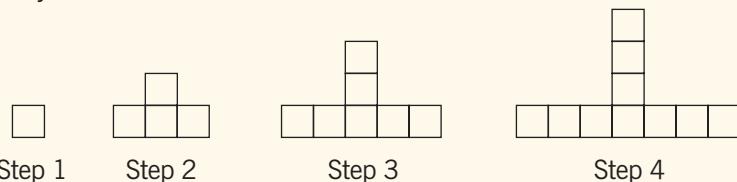
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Fig. 1 The Upside Down T Pattern task

Directions: Use the pattern below to answer the following questions. Please show all your work.



- Draw the next two steps in the Upside Down T Pattern.
- How many total tiles (i.e., squares) are in step 5? Step 6?
- Make some observations about the Upside Down T Pattern that could help you describe larger steps.
- Sketch and describe two steps in the pattern that are larger than the 10th step.
- Describe a method for finding the total number of tiles in the 50th step.
- Write a rule to predict the total number of tiles for any step. Explain how your rule relates to the pattern.
- Write a different rule to predict the total number of tiles for any step. Explain how your rule relates to the pattern.

nent in developing an understanding of function (Knuth 2000).

In addition to their potential to help students develop algebraic reasoning, pattern tasks may also be a powerful tool in helping establish classroom norms and practices at the beginning of the school year. In this article, we consider the mathematical and social purposes that pattern tasks can serve by examining one middle school's pattern revolution—a unit on patterns that is used to launch each school year.

A PATTERN REVOLUTION

During the first few weeks of each school year, students in the seventh and eighth grades at Bellfield Middle School explore a series of geometric pattern tasks. Teachers work together to identify tasks from a variety of sources (e.g., journals, materials from teachers' master's-level course work and professional development experiences, and other curricular materials) and decide which tasks to use at each grade level to ensure that students explore different ones each year. Even with these departmental decisions,

teachers have flexibility in determining the amount of time spent on the unit (usually one to two weeks) and in selecting and sequencing the pattern tasks based on the needs of students. Beginning the school year with a unit on patterns has a number of advantages for students, both with respect to developing their capacity to reason algebraically and to participate in a learning community. These advantages are explored in the sections that follow.

PATTERN TASKS ACCESSIBLE TO ALL

All students, regardless of prior knowledge and experiences, can explore pattern tasks. For example, in solving the Upside Down T Pattern task (shown in **fig. 1**), some students may build subsequent steps using square tiles or draw the next steps on grid paper, some may make a table and look for numeric patterns, some may view the pattern in one of the ways shown in **figure 2**, or others may simply notice the recursive "plus 3" pattern. Hence, all students can do something mathematical when presented with a

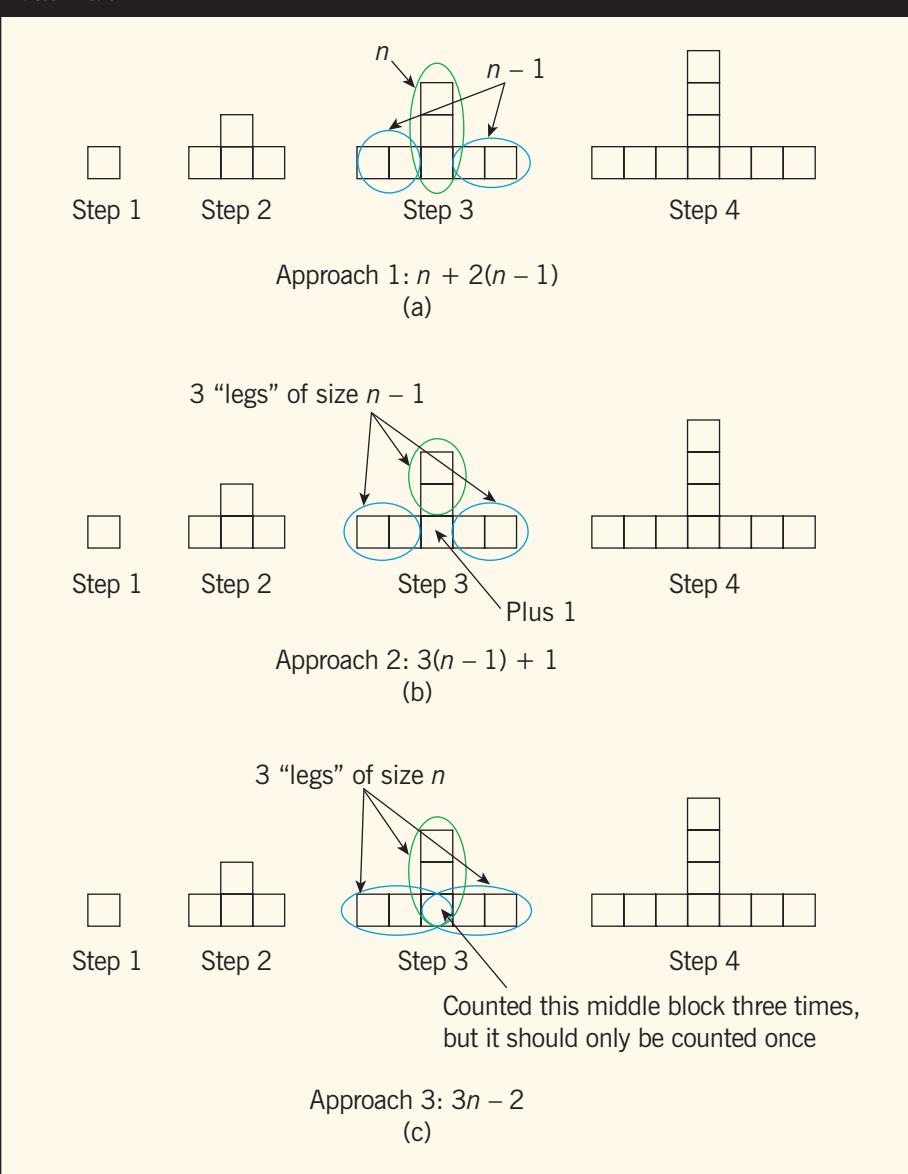
geometric pattern. One teacher noted that regardless of your background, you can fly into the task anywhere. You can have the brightest kid in your class *and* the one who is struggling feel success from the first two weeks. "So it makes everybody feel like they're on kind of an even playing ground." Once a student has a foothold on solving the task, the teacher is then positioned to ask questions to assess what the student understands about the relationships in the task and to advance students beyond their starting point.

In addition, students who have opportunities to explore and discuss relatively simple linear patterns, such as the Upside Down T Pattern, will be able to draw on their work as they encounter more complex items, such as the S Pattern task (shown in **fig. 3**), which grows in two directions. For example, students who created solutions such as those in **figure 2a** or **2b** (in which the Upside Down T was broken into different chunks) might try a similar strategy for the S Pattern (e.g., breaking the S into a top row of size $(n + 1)$; a bottom row of size $(n + 1)$; and a rectangle, or square, with dimensions $n \times n$). Since strategies are shared publicly, all students have opportunities to make sense of and draw on a range of different strategies in subsequent work.

TASKS CAN BE REVISITED

Pattern tasks that are explored at the beginning of the school year can become memorable for students and can be referenced as the year progresses. In so doing, these tasks further develop students' understandings of key algebraic ideas. For example, once students have some experience graphing linear equations in two variables, students could draw a graph that shows the number of tiles in each step as a function of the step number in the Upside Down T Pattern task. Students might then consider how

Fig. 2 Three visual approaches for determining a generalization for the Upside Down T Pattern task



the "plus 3" pattern they noticed at the beginning of the year relates to the graph—thus making important connections between the change in successive steps in the pattern, the rate of change in a linear relationship, and the slope of the graph. Alternatively, if students have explored both linear and nonlinear growth patterns (such as the Upside Down T Pattern shown in **fig. 1** and the S Pattern shown in **fig. 3**), they might be asked to graph both and to compare and contrast the graphs and the expressions. This could

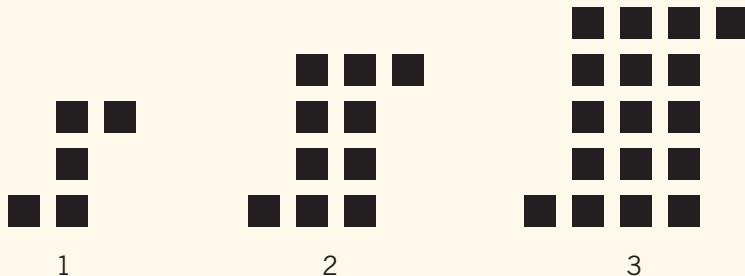
lead to a discussion about how the underlying nature of the pattern (i.e., the growth rate) helps predict whether or not the graph is linear.

A CONTEXT FOR DISCUSSING MULTIPLE SOLUTION STRATEGIES

There are usually many different, yet equivalent, ways of expressing the relationship between two variables in a pattern task. For example, in solving the Upside Down T Pattern task, students might determine at least

Fig. 3 The S Pattern task

The first three figures in a pattern of tiles are shown below.



ADAPTED FROM THE MATH LEARNING CENTER 1996

the teacher values different ways of thinking about the same problem, and that solving a task in one way is not sufficient. From the teacher's perspective, solving tasks in multiple ways also encourages students to become flexible in their thinking. This skill will benefit students as they encounter problems beyond pattern tasks.

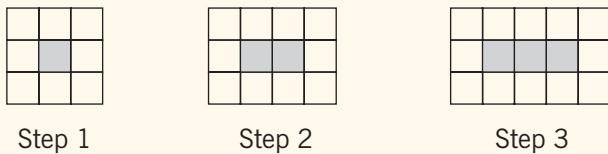
DEVELOPING CLASSROOM NORMS AND PRACTICES

Generating and sharing multiple solutions also provides a way to help students learn how to participate in the classroom community. For example, instead of the teacher beginning the school year by giving students a list of classroom rules, students learn important norms and practices through participation in a community where the practices are reinforced as students engage in mathematical work. For example, beginning on the first day of the pattern unit (usually the first day of school), students are assigned to groups and encouraged to both talk with and listen to one another. Students are consistently asked to look for alternative ways to solve problems and to question the teacher and one another when they do not understand. Students realize that the teacher is a source of support and encouragement but is not someone who is there just to give out answers.

Sharing solutions also serves to establish key aspects of classroom culture. To illustrate how norms and practices such as accountability, clarity, and respect might be established, we zoom in on a classroom in which students are sharing solutions to the Garden Pattern task (shown in **fig. 4**). The students worked on the task in their small groups for about fifteen minutes and then participated in a fifteen-minute whole-class discussion that served two purposes: (1) to determine the extent to which there was consensus on the answers

Fig. 4 The Garden Pattern task

Study the pattern below of forming a row of black squares surrounded by white squares. As you answer the questions, record any patterns you notice.



Step 1

Step 2

Step 3

ADAPTED FROM FRIEL, RACHLIN, AND DOYLE 2001

- Assume the pattern continues, and draw the next step in the pattern.
- How many white squares will be in the 5th step?
- If there are 7 black squares in a row, how many white squares will there be?
- How many white squares will surround 50 black squares?
- How many black squares will be in the row if there are 100 white squares surrounding them?
- How many black squares will be in the row if there are 71 white squares surrounding them?
- Generalize the number of white and black squares for any step (write a rule, make an equation, state a fact using a step, and so on).

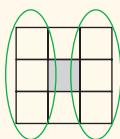
three different symbolic ways of expressing the relationship between the step number and the number of tiles in each step (shown in **fig. 2**). From a mathematical perspective, this also provides an opportunity to discuss why these expressions are equivalent and how you could justify it. For example, in justifying why the expressions that generalize the relationship in the Upside Down T Pattern

$$n + 2(n - 1), 3(n - 1) + 1, \text{ and } 3n - 2$$

are equivalent, students would have opportunities to use algebraic skills, such as combining like terms, using the distributive property, and substituting values for n in each expression and determining whether all the expressions produce the same answer.

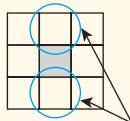
In addition, as students explore pattern tasks individually and in small groups, the teacher can challenge them to determine different ways of viewing, describing, and generalizing the pattern. Thus, students learn that

Fig. 5 Beth's approach to the Garden Pattern task



Step 1

Beth's "add six" approach
(a)



These two portions have the same amount of squares as the black squares in each step

The "two times the number of black squares" in Beth's approach
(b)

to parts a, b, c, and d; and (2) to make public different ways of determining the number of white squares, given any number of black squares. Beth is the first student to go to the overhead projector and share her thinking about how she determined the number of white squares in each step.

Teacher: OK, Beth, go ahead, tell us how you figured, and everybody pay attention.

Beth: [She walks to the overhead projector, which contains a transparency of the first three steps in the pattern.] You multiply by two and add six.

Teacher: You multiply *what* by two?

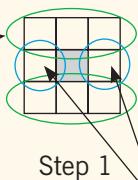
Beth: The black squares.

Teacher: Write it down, somewhere. On the top or something. What is she going to multiply by two, Adam?

Adam: [He goes to the overhead projector and writes, "Multiply black squares by 2, add 6."]

Fig. 6 Faith's approach to the Garden Pattern task

1. "I did the number of black squares and I added two."



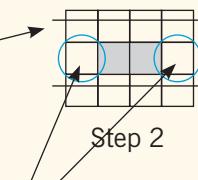
Step 1

2. "And then you do that times two. The three squares up there, times two, equals these three squares down here."

3. "Plus two."

Faith's explanation of her approach for step 1
(a)

1. "Since there's two black squares, then, you would add two to them. So, then, two plus two would be four squares up here."



Step 2

2. "And then, you would do four times two, and that's eight. And then, that would be that, down here."

3. "And then, you would add two to eight, for these two squares right here, and that would equal ten."

Caitlyn's explanation of Faith's approach for step 2
(b)

Teacher: OK, what did he add there?

Does anybody see the little scribble? What did he say?

Sara: Black squares.

Teacher: Black squares. You multiply the black squares by two, and then add six. Can you show us on the diagram? Where do you see it on the picture? Where do you see that, to multiply by two? Show on there. Show on the picture. You can write on it [the transparency].

Beth: [She demonstrates her method on step 1, **fig. 5**.] There's one, then one square times two equals two, plus six, equals eight, and then, it's eight squares.

Teacher: OK, you add six. Where is the constant of six?

Beth: Because there's three on each side.

Teacher: Circle them for me.

Beth: [She makes circles around the squares on the sides of step 1, as shown in **fig. 5a**.]

Teacher: One, and the two—where's the two? Two ones, are where?

Beth: Right there, and right there [points to the middle square of the three squares on the top row and the bottom row of step 1, as shown in **fig. 5b**.]

After Beth's presentation, the teacher presses students to express Beth's way of viewing the pattern symbolically as $y = 2b + 6$, where b is the number of black squares and y is the number of white squares. The teacher then solicits a second method from the class, and Faith volunteers to share her thinking. Faith describes her approach on step 1 (shown in **fig. 6a**), which makes use of chunking the garden

pattern into top and bottom rows that each contain $n + 2$ white squares, plus two white squares (one to the left of the black square and one to the right of the black square). When she is finished with her explanation, the teacher comments, "OK, I don't think they [the students in the class] understood that." Faith describes her approach about how she determined the number of white squares in step 1 again, and the teacher questions Faith and others about her approach.

Teacher: Where's your plus two? Show the class where plus two is.

Faith: These two right here [points to the white squares to the left and right of the black square in step 1], because they're the two remaining squares that you haven't added already.

Teacher: OK, how about the next one? Did anybody pick that up? Can anybody do the next one? All right, so Caitlyn, so show us on the second one. See if you understand what Faith was doing.

Caitlyn: Oh. Oh, yeah.

Teacher: Draw on it. Mess it up. Go ahead.

Caitlyn: [She walks to the overhead projector and demonstrates Faith's method on step 2.] Since there's two black squares, then, you would add two to them. So, then, two plus two would be four squares up here [draws a line across the four white squares that make up the top of step 2, shown in fig. 6b].

Teacher: OK.

Caitlyn: And then, you would do four times two, and that's eight. And then, that would be that, down here [draws a line across the four white squares that make up the bottom of

step 2]. And then, you would add two to eight, for these two squares right here [points to the white squares to the left and right of the two black squares], and that would equal ten.

The teacher then demonstrates Faith's way on the third step of the pattern. At the end of class, Phoebe presents a third rule for finding the number of black and white squares in the Garden Pattern.

Teacher: Did anybody else come up with any other strategies?

Phoebe: OK, well, mine's kind of confusing. It's kind of like Faith's but not. . .

Teacher: OK. Well, say it.

Phoebe: [She goes to the overhead projector and uses step 2 to demonstrate her method, shown in fig. 7.] OK, there's always going to be two more squares on the bottom [row].

Teacher: Draw on it.

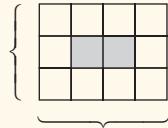
Phoebe: There's always going to be two more squares down here, than there is right here. So, I knew that—this was for the fifty [black squares] one [question d in fig. 4].

Teacher: This is for the fifty [black squares] one [question d in fig. 4]. OK.

Phoebe: So, I knew that there was going to be fifty-two on the bottom, for the fifty problem, 'cause there's fifty black squares. And, so I took fifty-two times three, these three, 'cause there's always three on the side, and I got a hundred fifty-six. And then, since there's fifty, which gives you the area, and since there's fifty colored in, then you have to subtract that by fifty, and that gives you a hundred and six.

Fig. 7 Phoebe's approach to the Garden Pattern task

"There's always three on the side."



"There's always going to be two more squares on the bottom [row]."

Multiplying the length of the bottom row by three gives you the area. You have to subtract that by the number of black squares.

Teacher: Oh! That was pretty creative. She took the whole figure, and then subtracted out the area in the middle. Oooh, I like it.

Several aspects of these excerpts are noteworthy. First, students explain and clarify their thinking with the help of comments and questions posed by the teacher. For example, in the beginning of her explanation, Beth tells the class to "multiply by two and add six." The teacher then asks, "You multiply *what* by two?"—a question that helped to clarify Beth's initial statement. In addition, after Faith demonstrates her method on step 1, the teacher presses her for further clarification by commenting, "OK, I don't think they [the students in the class] understood that." With help from the teacher, Faith then provides an explanation that is more clearly connected to the pattern. The teacher also ensures that a clear written record of students' thinking is created by asking students to draw and record their ideas on the transparency. Through these moves, students also learn which aspects of their explanations should be recorded so that everyone in the class can understand the presented strategy.

In addition, students in the class

are held accountable for making sense of solutions presented by others. For example, during Beth's presentation, the teacher asks other students to help clarify statements made by Beth and Adam. After Faith demonstrates her method on the first step of the pattern, the teacher holds students accountable by asking if anyone can apply Faith's strategy to find the number of squares in the next step in the pattern. In response to this query, Caitlyn explains Faith's method on the second step of the pattern. Thus, students learn that they are responsible not only for explaining their own thinking but also for carefully listening to, making sense of, and explaining the ideas shared by other students in the class.

During this discussion, the class is also developing important norms regarding respect. The teacher shows respect for her students' mathematical abilities by positioning them as authors of mathematical work. For example, when Faith offers an explanation that the teacher feels may be unclear to other students in the class, the teacher presses Faith to continue her explanation, instead of taking over the explanation for her. Students also show respect for one another's thinking by applying other students' solution strategies.

Finally, important mathematical ideas are publicly made during the discussion. For example, the solution strategies shared by Beth, Faith, and Phoebe are *explicit* forms of pattern generalization (Friel, Rachlin, and Doyle 2001) (rather than *recursive* strategies). That is, the students' generalizations represent the relationship between the step number and the number of white tiles such that the input is related to the output by a well-defined functional rule (Driscoll 1999). For example, Beth notes that "you multiply [the number of black squares] by two and add six."

In addition, students are encouraged to make connections between representations—an activity that helps students make sense of relationships such as linear functions (Friel, Rachlin, and Doyle 2001) and develop a more robust understanding of functions (Knuth 2000). In their work on the Garden Pattern task, the teacher presses students to make connections between their generalizations and the geometry of the pattern.

CONCLUSION

As we have demonstrated in the article, pattern tasks can provide rich opportunities for students to begin to explore relationships between variables and patterns of change. In addition, as one teacher noted, pattern tasks can help set the tone for the school year by helping students "get used to how you run your classes that might be different from other math experiences." It is also important to note that students at Bellfield Middle School appear to recognize the benefits of the pattern unit. For example, one student noted, "Starting with [the pattern unit] made me feel more comfortable in class. I know I can ask questions and not feel stupid. Now I feel encouraged to participate and I enjoy math more." Another student commented, "I was never good at math and I always started out dreading math class. This year I have new hope because I'm allowed to think differently and I'm having success. I look forward to coming to class every day now."

So how does one start such a pattern revolution? At Bellfield Middle School, the pattern revolution began with one teacher sharing her students' work on geometric pattern tasks with colleagues. This student work provided an opportunity for teachers to have grounded conversations about what students were able to do mathematically. Other teachers were intrigued by the pattern tasks, and one by one

began asking the teacher to share them. As the pattern unit has evolved over time, teachers have had opportunities to draw on a shared experience that has served as the basis for conversations in the teachers' lounge and at department meetings, as well as a platform for further collaboration and discussion.

It is interesting to note that the teachers at Bellfield Middle School have recently adopted a reform-oriented curriculum—due in part, they claim, to their opportunities to witness firsthand what students can do mathematically when posed with high-level tasks during the pattern unit. Despite the adoption of the new curriculum, teachers continue to launch the school year with the pattern unit, since they feel that it serves important mathematical and social purposes.

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