Traditionally, high school algebra instruction has emphasized manipulating symbols to solve equations and simplify expressions. As such, algebra courses have often played the role of a sorter and sifter of students rather than that of a pump that engages all students in empowering mathematical reasoning (National Research Council 1989). In contrast to the traditional role of school algebra as following procedures for manipulating symbols, *Principles and Standards for School Mathematics* recommends that students in grades K–12—

- understand patterns, relations, and functions;
- represent and analyze mathematical situations and structures using algebraic symbols;
- use mathematical models to represent and understand quantitative relationships; and
- analyze change in various contexts. (NCTM 2000, p. 37)

The expectations in the Algebra Standard are that students in grades 9–12 will

- generalize patterns using explicitly defined and recursively defined functions; . . . use symbolic algebra to represent and explain mathematical relationships; . . . [and] use symbolic expressions, including itera-
tive and recursive forms, to represent relationships arising from various contexts. (NCTM 2000, p. 296)

These expectations relate to two important instructional goals for high school students: first, that they can reason flexibly, using recursive and explicit reasoning when faced with the need to create a mathematical model for a situation; and second, that they recognize the advantages and limitations of these two ways of reasoning. In the following sections, I provide an initial discussion of the use of recursive and explicit reasoning, elaborate on the types of tasks that encourage students to reflect on these advantages and limitations, and describe some important considerations for classroom discussion when using these various tasks.

BACKGROUND ON THE USE OF RECURSIVE AND EXPLICIT REASONING

Students naturally reason recursively when they begin to examine patterns. Recursive reasoning uses an established mathematical relationship between a previous term or terms in a sequence. For example, in the Beam-Design problem in figure 1, students often notice that when the beam length increases by one unit, four more rods must be added onto one end of the diagram. This rule can be written with formal mathematical symbols as \( u_n = u_{n-1} + 4, \) or with a less formal notation currently employed in some mathematics textbooks as NEXT = NOW + 4, Start = 3.

In addition to creating a recursive rule, students can construct a number of explicit rules for the Beam-Design problem. The following are three common approaches that students often use for this task.

Method 1
This strategy builds on the recursive reasoning discussed previously; it recognizes that \( n - 1 \) groups of four rods are added to the initial length-one beam containing three rods, as depicted in figure 2a. It leads to the explicit rule \( R = 3 + 4(n - 1) \), where \( n \) is the beam length and \( R \) is the number of rods.

Method 2
Another explicit rule for this situation is \( R = 4n - 1 \), where the expression \( 4n \) counts the number of rods by using \( n \) groups of four rods and subtracts 1 for the rod missing from the leftmost section of the beam, as illustrated in figure 2b.

Method 3
A third rule that students often determine is \( R = 3n + (n - 1) \), in which the number of rods on the triangles is counted by using the expression \( 3n \) for each group of three rods on the triangles at the bottom of figure 2c. The remaining number of rods at the top of the beam requires \( n - 1 \) rods. This rule gives the number of rods needed for beams of length 4, 5, 10, and 76 as 3, 7, 22, and 91, respectively.

Beams are designed as a support for various bridges. The beams are constructed using rods. The length of the beam is determined by the number of rods used to construct the bottom of the beam. Below is a beam of length 4.

1. How many rods are needed to make a beam of length 5? Of length 10? Of length 20? Of length 76?
2. Write a rule or a formula for finding the number of rods needed to make a beam of any length. Explain your rule or formula.

Fig. 1 The Beam-Design problem

![Fig. 1](image1)

![Fig. 2](image2)

![Fig. 2](image3)
the top of the diagram is \( n - 1 \), since there is one fewer rod on the top of the diagram than there are triangles at the bottom of the diagram.

All these methods generate equivalent rules but represent different ways of viewing the problem situation. Historically, explicit reasoning has been valued over recursive reasoning because of the inefficiency of repeatedly performing calculations (that is, iterating the function), especially when these calculations are performed by hand. However, with the wide range of technological tools currently available, using recursive rules is no longer the tedious exercise that it once was. For example, students can quickly generate the recursive rule for the Beam-Design problem on a computer spreadsheet. In figure 3, the leftmost table displays the values that the spreadsheet shows, and the right side contains the corresponding formulas that underlie the values generated on the left side of the figure.

Traditionally, algebra courses have used only those tasks that students can model explicitly. The situations that students can model are then limited and the introduction of everyday mathematical situations is delayed because students have not been introduced to the sophisticated mathematical ideas necessary to derive an explicit rule. An example occurs with an investment situation involving ordinary annuities, given in figure 4. For this situation, the recursive rule

\[
\begin{align*}
  u_n &= \frac{1 + 0.05}{12} u_{n-1} + 100, \\
  u_0 &= 3000,
\end{align*}
\]

where \( u_n \) represents the amount of money in the account after \( n \) months, is relatively easily determined. However, deriving an explicit rule for this situation involves using the algebraic identity

\[
1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1},
\]

an identity that is unfamiliar to beginning algebra students. Although students are often given an explicit formula and asked to use it to model this situation, they often memorize the formula with little

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Fig. 3 Recursive table for the Beam-Design problem
Suppose that you have $3000 in a savings account that earns 5 percent annually. In addition, you deposit $100 into the account each month. Determine the amount of money in your account after \( n \) months.

**Fig. 4** The investment problem

understanding and therefore easily forget it. However, most high school students can model this situation if they are allowed to represent the situation with a recursive rule.

**TYPES OF ALGEBRAIC TASKS**

Patterning tasks can be classified into three broad categories:

- Those that students often choose to model recursively and explicitly (flexible tasks)
- Those that students prefer to model explicitly
- Those that students prefer to model recursively

For each type of task, the teacher should encourage students to examine a number of specific instances for the situation before generating a general rule. Additionally, requiring students to record their thinking in a journal or as notes in their notebooks as they progress through various instances can help the teacher better assess the difficulties and understanding that each student has. Students often change strategies while they continue to examine particular instances, moving back and forth between explicit and recursive strategies.

**Flexible tasks**

When students are introduced to algebraic modeling, beginning with flexible tasks is often beneficial. Flexible tasks are those for which both the recursive relationship and the explicit relationship are evident. Students can look at a flexible task and easily determine the change from one instance to the next or think about a general relationship that exists across many cases. These tasks often bring out both recursive and explicit rules from students.

Tasks such as the Beam-Design problem and the Phone-Cost problem, in figure 5, allow students to reason recursively or explicitly. In the Phone-Cost problem, the recursive rule for the cost of the telephone call over five minutes is \( u_n = u_{n-1} + 6, \ u_5 = 50 \).

Determining an explicit rule for the cost of a telephone call over five minutes involves thinking about how many times 6 is added to the cost of the first five minutes (50 cents). One way that students often determine an explicit rule for this situation is through recognizing that the number of times that 6 is added is equal to the number of minutes by which the telephone call exceeds the initial five minutes.

The Jog Phone Company is currently offering a calling plan that charges 10 cents per minute for the first five minutes of any telephone call. Any additional minutes cost only 6 cents per minute.


2. Explain how you would determine the cost for any phone call that is 5 minutes long or longer. Write a rule to explain how you would determine this cost.

**Fig. 5** The Phone-Cost problem

This reasoning can result in the rule \( C = 6(n - 5) + 50 \), for \( n > 5 \). Explicit and recursive reasoning are intertwined in the construction of the explicit rule.

After students have generated a variety of recursive and explicit rules, I encourage them to reflect on their strategies. Questions that I ask include the following: Which rule is preferable? What are the advantages of using an explicit rule? When do you prefer to use a recursive rule? When do you prefer to use an explicit rule? What connections exist among the recursive and explicit rules? Asking such questions encourages students to examine the mathematical power and utility of these different ways of reasoning.

The students can make additional connections about the relationship between the concept of slope and the recursive and explicit rules generated in this example. Asking students what the slope means in this situation and how it corresponds to the recursive and explicit rules can help them begin to see how this concept is represented in these rules. In the Phone-Cost problem, the slope is represented in the recursive rule as adding 6, corresponding to the increase in the cost of a telephone call as 6 cents per minute. The coefficient 6 in the explicit rule \( C = 6(n - 5) + 50 \) also results in an increase in the cost of the telephone call of 6 cents per minute.

**Explicit-preferred tasks**

Following the use of flexible tasks, it is important to use explicit-preferred tasks, those tasks that students primarily model by using explicit rules. One benefit of explicit-preferred tasks is that they can encourage students who prefer to use only recursive thinking to examine the power of explicit reasoning. For explicit-preferred tasks, students tend to develop explicit rules because the recursive relationship is not evident or because the recursive relationship varies in such a way that it cannot be easily determined. The Border problem, given in figure 6, is an example of a task in which the recursive relationship is hidden. Rarely will students

Vol. 98, No. 4 • November 2004 | MATHEMATICS TEACHER 219
construct a recursive rule for this situation, although they are generally able to create a variety of explicit rules.

One explicit rule that students often generate involves noticing that four groups of \( n \) patterned squares are on the border of the diagram and four extra squares are on the corners, as shown in figure 7a, leading to the explicit rule \( B = 4n + 4 \), where \( n \) is the width of the border and \( B \) is the total number of patterned squares. Additionally, students often recognize the four groups of \( n + 1 \) patterned squares arranged around the border, as shown in figure 7b, and construct the explicit rule \( B = 4(n + 1) \). Other rules that students often derive include \( B = 2n + 2(n + 2) \), shown in figure 7c, pairing the corners with the top and bottom sides of the border, and \( B = (n + 2)^2 - n^2 \), subtracting the number of squares in the inner portion of the diagram from the total number of squares, shown in figure 7d.

After students have explained their various rules, the teacher should continue to ask them to determine the advantages and disadvantages of the explicit strategies generated in the classroom. In addition, I ask students to determine the recursive rule for this situation. When most students model the situation, the recursive relationship remains in the background. To encourage students to examine it further, I ask them what the slope means in the explicit rule \( B = 4n + 4 \) in relation to the situation and why the recursive rule that models this situation, \( u_{n+1} + 4, u_1 = 8 \), increases by 4 each time. This explanation presents a challenge for students, since they have difficulty relating these ideas to the context of the problem. Instead, they typically describe the increase in the results of their explicit rule (for example, when \( n = 2, 3, \) and 4, the number of squares is 12, 16, and 20, respectively). It is important to focus students back onto the context of the situation by asking how these ideas relate to constructing the border. With further thought, students may determine that a border of length \( n \) can be constructed from the border of length \( n - 1 \) by inserting a square into each of the four sides in the diagram. The use of square tiles or some other type of manipulative helps students recognize this relationship and communicate this idea effectively.

**Recursive-preferred tasks**

Recursive-preferred tasks are generally the final type of task that I give to students. Such tasks encourage students to realize that determining an explicit rule for a problem situation is often difficult. The book *Discrete Dynamical Modeling* (Sandefur 1993) includes many situations that are accessible to high school students. One such situation is the Pollution-in-the-Pond problem, given in figure 8. An explicit rule for this situation can be derived, although understanding the derivation is rather sophisticated for introductory algebra students. However, given time to grapple with the situation, students can use a recursive rule to model the amount of pollution in the problem. In this case, the recursive rule \( u_n = 0.5u_{n-1} + 5, u_0 = 20 \), where \( u_n \) represents the amount of pollution in the pond after \( n \) weeks, demonstrates that half the pollution remains in the pond each week and that five extra pounds are accumulated in the pond in this same time period. (I invite readers to consider how an explicit rule can be derived for this situation.) Again, asking questions that encourage students to reflect on their strategies is important. These questions include the following: Can you generate an explicit rule for this situation? Why or why not? How does this situation compare with others that you have examined? Why were you unable to generate an explicit rule? Is constructing an explicit rule always possible?

**IMPORTANT ISSUES FOR DISCUSSION**

As students begin to model the various types of situations mentioned previously, several essential concepts emerge that deserve further discussion. In the following sections, I discuss four important issues that merit attention:

- The advantages, limitations, and connections of recursive and explicit reasoning
- The meaning and use of variables
- Equivalent expressions
- Justification
ADVANTAGES, LIMITATIONS, AND CONNECTIONS WITH RECURSIVE AND EXPLICIT REASONING

As mentioned previously, it is important to encourage students to examine the advantages and disadvantages of recursive and explicit rules, the connections between these two types of reasoning, and the connections with other important mathematical ideas. Whenever students generate both explicit and recursive models for a situation, we should ask how these rules are related and ask them the advantages and limitations of these rules. The recursive rule is always closely linked to the slope (that is, the rate of change) of the function. The recursive rule describes the rate of change from one instance to the next. Asking students how the slope can be

As an employee for the Environmental Protection Agency, you have been asked to examine the pollution level in Stinky Pond. This pond holds 50,000 gallons of water and contains 20 pounds of pollution. The pollution is dissolved in the pond and is evenly distributed throughout. Each week, half of the polluted water pours out of the pond and is replaced by the equivalent amount of rainwater containing 5 pounds of pollution.

1. How much pollution remains in the pond after 5 weeks? After 12 weeks? How much pollution remains in the pond after 267 weeks?

2. Write a rule or formula to determine how much pollution remains in the pond after any number of weeks. Explain your rule or formula.
determined from this rule—whether the rule is recursive or explicit—focuses students’ thinking on the differences in the ways that slope is represented for these two models. Additionally, asking what the slope means in this situation is also essential. Too often, students can provide the value for the slope but cannot relate this value to the problem context. Students should describe slope as a rate of change (for example, 6 cents per minute or 4 squares per 1 unit of increase in the length of the border). Discussing these ideas encourages students to examine the rate of change in a variety of situations. In addition, these ideas lay the foundation for future discussions in calculus.

Another way to encourage students to consider the connections between the concept of slope and the recursive rule is to ask them to find output values for consecutive values of the domain. For example, in the Phone-Cost problem, I ask students to determine the cost of a telephone call that is 37 minutes long and then find the cost of a 38-minute telephone call. I next ask students to describe the different ways that they determined these values. Generally, students will determine these values in one of two ways: they will use their explicit rule to find both the cost of a 37-minute call and a 38-minute call, or they will use their explicit rule to find the cost of a 37-minute telephone call and a recursive rule to determine the cost of the 38-minute telephone call. I then ask them which of these ways is preferable. A discussion follows that encourages students to reflect on the advantages and limitations of recursive and explicit reasoning in this situation. Students quickly recognize that the recursive rule is effective for computing the next value and that the explicit rule allows for quick computation of “farther” values of the independent variable. Without discussions about the efficiency of these two ways of reasoning, students often reason inflexibly in this type of situation—continuing to reason exclusively recursively or explicitly without understanding the power of these types of reasoning.

The meaning and use of variables

Another important consideration is the meaning of variable that students are developing through these activities. Students should be asked to explain what the variable represents and to describe appropriate values in the domain for their explicit rules. For example, after students generate the rule $B = 4n + 4$ for the Border problem in figure 6, they should discuss appropriate values that can be used for $n$. Asking what values can be used for $n$ in this situation encourages students to examine the possible domain values. Examining these values is often an overlooked aspect of this type of activity. Without further discussion about the possible values that the variable represents, some students may continue to view variables as representing only a single value. Identifying the domain of the variable raises important questions about whether the explicit model properly models the situation for all the values in the domain—an issue that I discuss in more detail in the justification section of this article.

Whether recursive or explicit models are used, students need to understand whether the variables used are discrete or continuous. Explicit rules can be used to quickly determine values for continuous variables, whereas recursive rules limit the calculations to discrete instances of the situation. For example, in the Phone-Cost problem, in figure 5, the time of the telephone call is a continuous variable, but the recursive rule $u_n = u_{n-1} + 6, u_1 = 50$, allows only for calculations of the cost of the telephone call after precisely $n$ minutes, where $n$ is a whole number. In contrast, the explicit rule for the cost of a telephone call over five minutes, $C = 6(n - 5) + 50$, allows for calculating parts of a minute (for example, this explicit rule can be evaluated when $n = 2.5$). For this situation, however, the domain of the explicit rule is also, in practicality, discrete. For the telephone company to bill the customer, it must make a decision about the precision of the length of the telephone call, often rounding the length of the call up to the nearest minute or tenth of a minute.

For the Pollution-in-the-Pond problem, the recursive rule limits the time period in which we can examine the amount of pollution in the pond. The recursive model discussed previously, $u_n = 0.5u_{n-1} + 5, u_0 = 20$, where $u_n$ represents the amount of pollution in the pond after $n$ weeks, calculates the amount of pollution after a given number of weeks. However, if we can generate an explicit rule for this situation, we can use it to determine the amount of pollution after any given time period. When we use an explicit rule in a continuous manner, we make assumptions about the rate at which pollution and rainwater flow back into the pond—assuming that the rate for both of them is constant. In general, students should understand that for any situation, the recursive model is limited to discrete instances of the situation, whereas the explicit model can be used for continuous cases. However, the situation must dictate whether an explicit model generates incorrect or inappropriate results.

A final issue related to the nature of the variable for these two models is the difference in the use of variable notation for recursive and explicit rules. For example, students can compare the use of $n$ in $B = 4n + 4$ with the use of $u_n$ in the equation $u_n = u_{n-1} + 4, u_1 = 8$. Whereas $n$ in the first rule is a varying quantity for which we can pick any value in the domain of the function, the value for $u_n$ is determined on the basis of previous iterations of the
rule. Questioning students about the meaning of their notation is essential so that they can understand how variables are used in recursive and explicit rules. Asking how use of $u_n$ in the rule $u_n = u_{n-1} + 4$, $u_0 = 6$, differs from the use of $n$ in the rule $B = 4n + 4$ can bring these issues to the forefront of classroom discussions.

Equivalent expressions
The Border problem can also lead to further discussion about why the expressions $4n + 4$ and $4(n + 1)$ are equivalent. In my work with students, I have found that students can provide powerful general arguments to justify the equivalence of these expressions. For example, one student explained that $4n + 4$ is “four more than $4n$” and that “$4(n + 1)$ also adds one more 4 to $4n$ since you are increasing the expression by multiplying by one more 4.” Such an argument relies on the strong connection that students have established between the recursive and explicit rules that they use. Students who are able to reason both recursively and explicitly in these situations are often able to provide such justifications regarding the equivalence of expressions. Without these connections and without understanding the relationship between multiplication and addition, students are often forced to mimic algebraic rules rather than understand the underlying reasoning that justifies these rules.

Justification
A final issue that is important to emphasize during discussions is related to the general nature of the models that students construct. It is essential that students understand that the explicit and recursive mathematical models that they create are generalizations that apply across all the cases in the domain for their functions. The teacher should ask students to justify why their rules correctly compute the desired quantity for any value in the domain. Such justification represents a considerable stumbling block for students and therefore deserves further attention in the classroom. Often students try to justify their rules by referring to specific instances (for example, “my rule works when $n = 2, 3,$ and $4$; therefore, it will always work”). Such attempts at justification are insufficient for a general rule and lack any connection that can help students construct a rule for a similar situation. Only when the rule is connected with a general relationship in the context can the rule be properly justified and explained in relation to the context of the situation. For examples of these types of generalizations, I refer back to my explanations of the rules $R = 4(n - 1) + 3$, $R = 4n - 1$, and $R = 3n + (n - 1)$ for the Beam-Design problem. For students to understand the general nature of the explicit and recursive rules that they construct, they must be asked to provide a sufficient general argument. Further discussion of the importance of justifying algebraic rules is provided in Lannin (2003).

CONCLUSION
Using a variety of tasks (flexible, explicit-preferred, and recursive-preferred) can foster the development of mathematical power while deepening students’ understanding of such important algebraic concepts as slope. It is important that we select tasks that encourage students to reason flexibly, both recursively and explicitly, and that they see the advantages and limitations of each type of reasoning. The mathematical power that results from such flexible reasoning allows students to model a variety of situations, connect important mathematical ideas, and build a basic understanding of algebraic concepts that can increase the mathematical success of all students.

REFERENCES

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